A DEVELOPMENT OF AN EXACT METHOD FOR THE SOLUTION OF SYMMETRICAL FIXED END PARABOLIC ARCHES

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by

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DEFINITION OF SYMBOLS

$\frac{M}{EI}$ diagram -- the moment diagram of a particular reaction or fixed end moment, divided by the modulus of elasticity, E, and also by the cross-sectional moment of inertia, I.

L -- length of span in feet.

h -- rise in feet--distance from springing line to crown.

w -- distance in feet from centerline to unit load.

a -- distance in feet from one end of span to unit load, measured along the springing line.

b -- distance in feet from other end of span to unit load, measured along the springing line.

H -- horizontal thrust in kips.

M_R -- Fixed end moment in kips at right springing.

M_L -- Fixed end moment in kips at left springing.

V_R -- vertical reaction at right springing.

V_L -- vertical reaction at left springing.

$K = \frac{L^2}{4h}$

x, y, z, -- coordinate variable distances, in feet.

$y = \frac{x^2}{K}$, equation of parabolic arch.

A -- area of $\frac{M}{EI}$ diagram of designating subscript.

M' -- first moment about centerline of arch of $\frac{M}{EI}$ diagram of designating subscript.

M" -- first moment of $\frac{M}{EI}$ diagram of designating subscript about a line through the vertex parallel to the springing line.

Designating subscripts:

p -- referring to composite $\frac{M}{EI}$ diagram of unit load and vertical reactions.

M_R -- referring to $\frac{M}{EI}$ diagram of fixed end moment at right springing.

M_L -- referring to $\frac{M}{EI}$ diagram of fixed end moment at left springing.

H -- referring to $\frac{M}{EI}$ diagram of horizontal thrust.

c -- distance in feet along X-axis from centerline of arch to centroid of $\frac{M}{EI}$ diagram of either fixed end moment.

d -- distance in feet along X-axis from centerline of arch to centroid of composite $\frac{M}{EI}$ diagram of unit load and vertical reactions.

m -- distance in feet along Y-axis from vertex to centroid of $\frac{M}{EI}$ diagram of either fixed end moment.

r -- distance in feet along Y-axis from vertex to centroid of composite $\frac{M}{EI}$ diagram of unit load and vertical reactions.

n -- distance in feet along Y-axis from vertex to centroid of $\frac{M}{EI}$ diagram of horizontal thrust.

$k = \frac{h}{L}$, ratio of rise to span.

$v = \frac{w}{L}$, ratio of unit load off-center distance to length of span.

C_1 to C_{15} -- constants.

I_x -- cross-sectional moment of inertia at any point along the arch.

I_c -- cross-sectional moment of inertia at center of arch.

α -- angle between tangent to working line of arch, and a line parallel to springing line.

ds -- increment of length along the arch itself.

dx -- increment of length parallel to X-axis (along the springing line).

$R = \sqrt{K^2 + L^2}$ $\qquad\qquad$ $N = \sqrt{\frac{K^2}{4} + w^2}$

OBJECT

The object of this thesis is to find an exact solution for the influence lines of the symmetrical fixed end parabolic arch, and to study the variations of these influence lines for arches of different spans and rises.

It is also the object of this thesis to compare the time required for the solution of the arch by the exact method mentioned above and by the accepted approximate methods.

Two general types of symmetrical fixed end arches will be considered: (1) those with a cross-sectional moment of inertia that is constant, and (2) those with a cross-sectional moment of inertia that varies according to some particular function.

INTRODUCTION

In attempting to find an exact solution for the parabolic arch, the principle of the conjugate structure will be employed. The conjugate structure method is a development of the first moment-area proposition. In the conjugate structure method, the real structure is laid on its side with the $\frac{M}{EI}$ diagrams of all the loads and reactions placed upon this re-oriented structure.

The first step is to find the areas of the various $\frac{M}{EI}$ diagrams, and to place them correctly upon the conjugate structure.

The second step is to find the location of the centroids of these areas.

The third step is to write the equilibrium equations for this conjugate structure, and to solve these equations for the three unknowns--horizontal thrust, moment at the right springing, and moment at the left springing.

To find the influence lines for the arch, each load point will be considered in succession. After determining the three unknowns mentioned above for each load point the influence lines can be drawn.

The first phase of this thesis deals with arches whose cross-sectional moment of inertia is constant. Exact formulas for the areas and the centroid distances of the $\frac{M}{EI}$ diagrams will be developed. Then, having found the values of these areas and centroid distances, these

quantities will be used in a succeeding set of equations that will give directly the exact values of the horizontal thrust, moment at the right springing, and moment at the left springing.

In addition to the general method just described, which can be used to solve any symmetrical parabolic arch having a constant moment of inertia, it will be shown that for arches having the same ratio of rise to span, (1) the horizontal thrust is the same, and (2) the value of the moments at the springing varies directly as the length of span.

The second and possibly more important phase of this thesis deals with parabolic arches whose cross-sectional moment of inertia varies according to some particular function.

This phase is considered to be more important than the first because of its wide application and the extreme simplicity of the method.

The method of development in this second phase is basically the same as that for the arch having a constant moment of inertia, but the results are a set of three simple equations which give directly the value of the thrust and the two springing moments for any symmetrical fixed end parabolic arch whose cross-sectional moment of inertia varies the same as that used in this paper. Even for varying moments of inertia that differ to quite some extent from the one chosen here, the results will be accurate to within a few percent of the true value.

PART I: CONSTANT MOMENT OF INERTIA

In order to fully understand the development of this solution, it will be necessary for the reader to understand the application of the conjugate structure. The conjugate structure, for a given real structure, is identical, in the lengths of its members and their relative positions, to the real structure. It is considered to be positioned, however, so that it is located in a horizontal plane with the loads of the $\frac{M}{EI}$ diagrams acting in a vertical direction.

If there is no slope or deflection at the supports of the real structure, there will be no shear or moment existing in the conjugate structure at those points. This latter condition obtains in the solution of the parabolic arch, so that the only loads or moments acting on the conjugate structure are those due to the $\frac{M}{EI}$ diagrams.

In order to place the $\frac{M}{EI}$ diagram correctly, the following convention will be adopted. Any load or moment in the real frame that causes tension on the inside surface thereof is considered to result in an $\frac{M}{EI}$ diagram acting down on the conjugate structure. Any load or moment that causes compression on the inside surface will have its $\frac{M}{EI}$ diagram acting upward.

After having placed the $\frac{M}{EI}$ diagrams upon the conjugate structure, and since there is no shear or moment at the ends of the conjugate structure, it will be held in

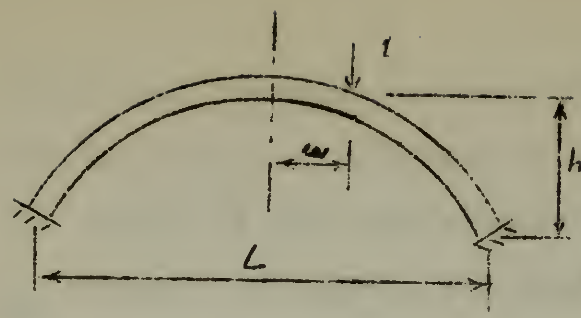


Fig 1

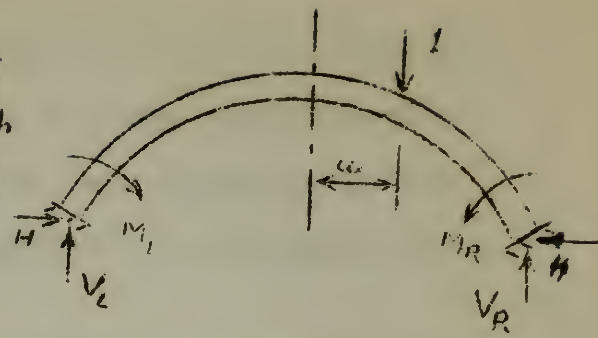


Fig. 2

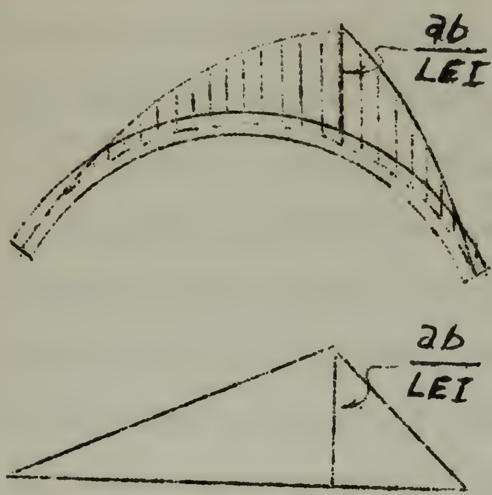


Fig. 3

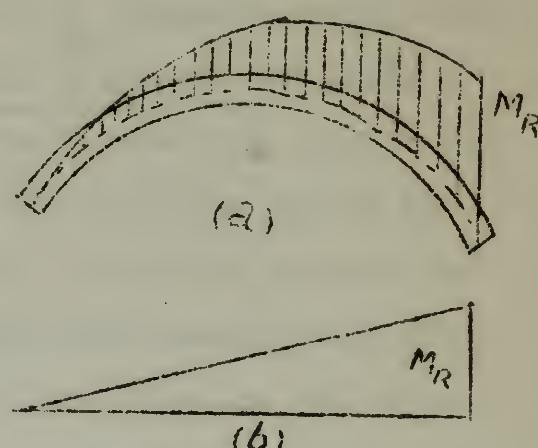


Fig. 4

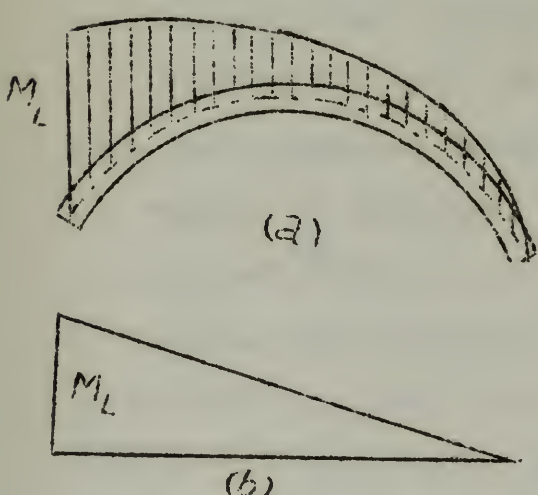


Fig. 5

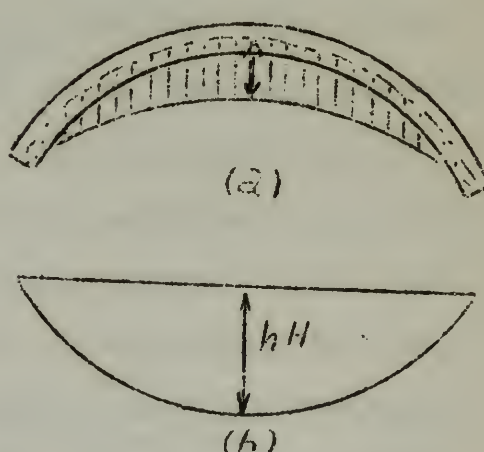
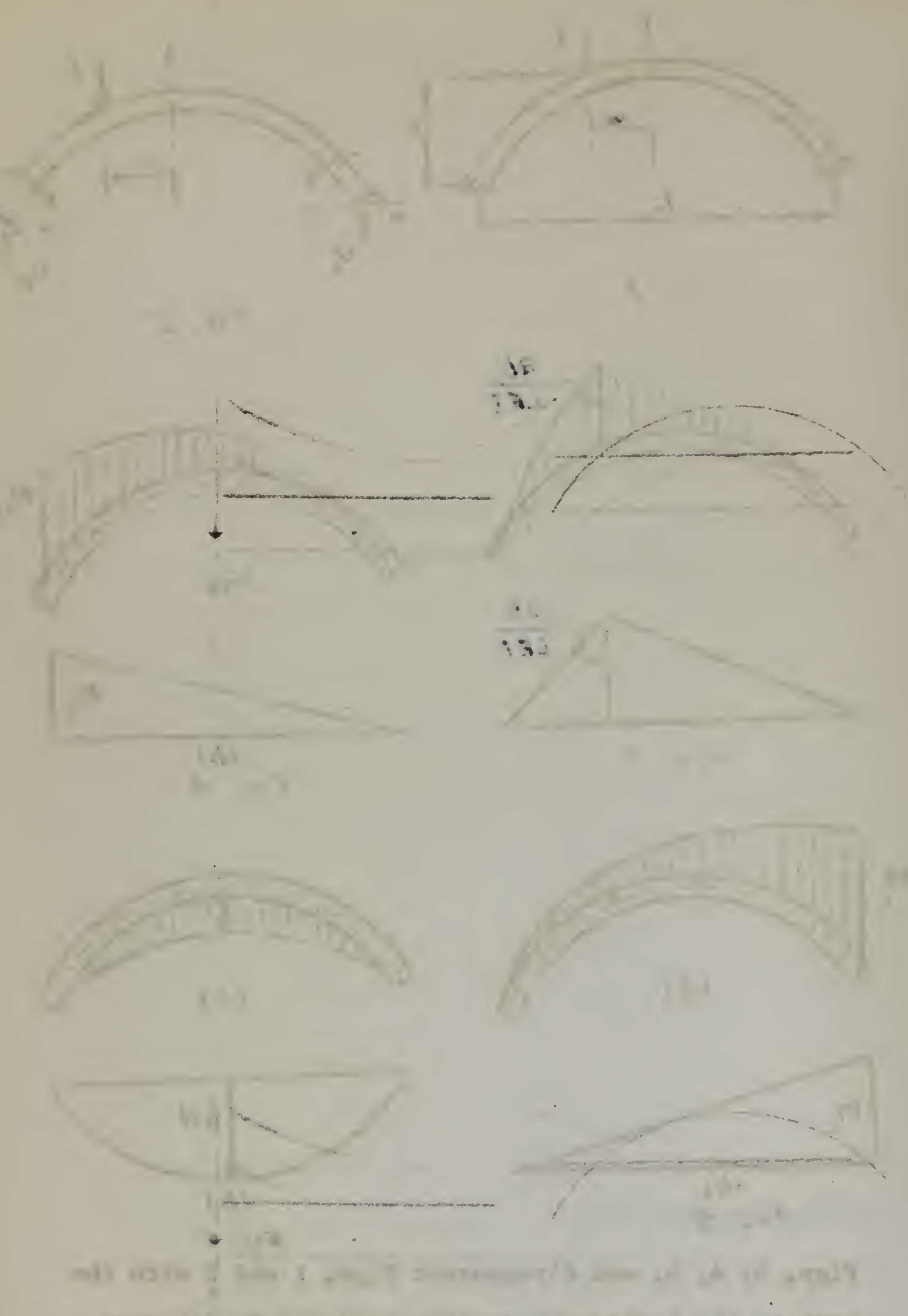


Fig 6

Figs. 3, 4, 5, and 6 represent Figs. 1 and 2 with the arch in the horizontal position and the $\frac{M}{EI}$ diagrams applied vertically.



Figs. 5, 6, 7, and 8 representing Figs. 1 and 2 with the load in the horizontal position and diagrams vertical position respectively.

always act downward on the conjugate structure. The horizontal thrust will always tend to cause compression on the inside of the real structure, and consequently its $\frac{M}{EI}$ diagram will always act upward on the conjugate structure.

At this point it will be assumed that both of the fixed end moments tend to cause tension on the inside of the real structure, resulting in $\frac{M}{EI}$ diagrams acting downward on the conjugate structure. Actually, for different load points across the span, these fixed end moments may tend to cause either tension or compression on the inside of the real frame. However, it will be pointed out later that by assuming both fixed end moments will tend to cause tension on the inside, the solution will automatically give the correct sign to be applied to the final value of these moments.

FINDING THE AREAS OF THE $\frac{M}{EI}$ DIAGRAMS. The first step in the general development is to find the areas of the various $\frac{M}{EI}$ diagrams. To do this the calculus will be employed. First, a set of three coordinate axes will be chosen, and a parabolic cylinder placed in the vertical position, with its vertex at the common intersection of the three axes. The equation of this parabolic cylinder is $y = \frac{x^2}{K}$ where $K = \frac{L^2}{4h}$, "L" being the length of span, and "h" being the rise of the arch above the springing line. To find the area of the diagram shown in Fig. 3, for the combination of the unit load and the

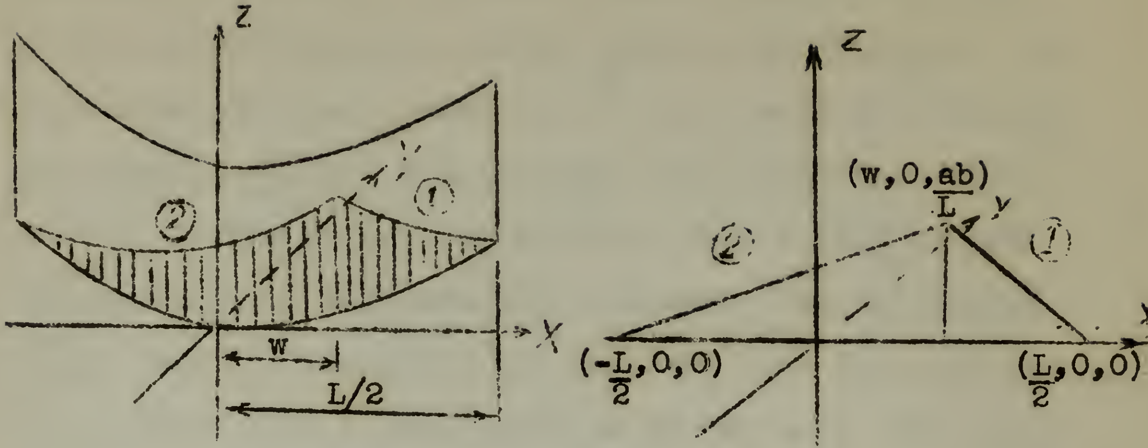


Fig.7

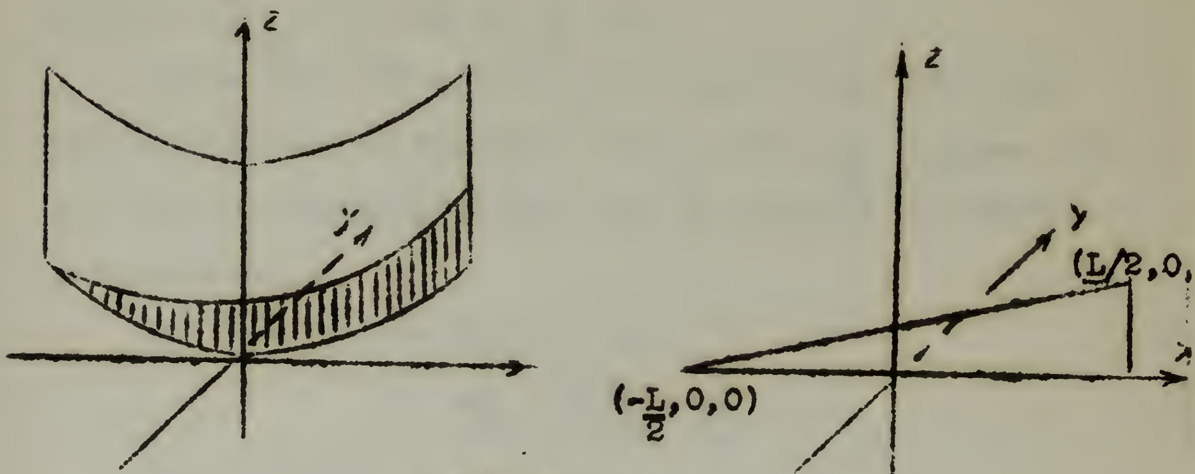


Fig. 8

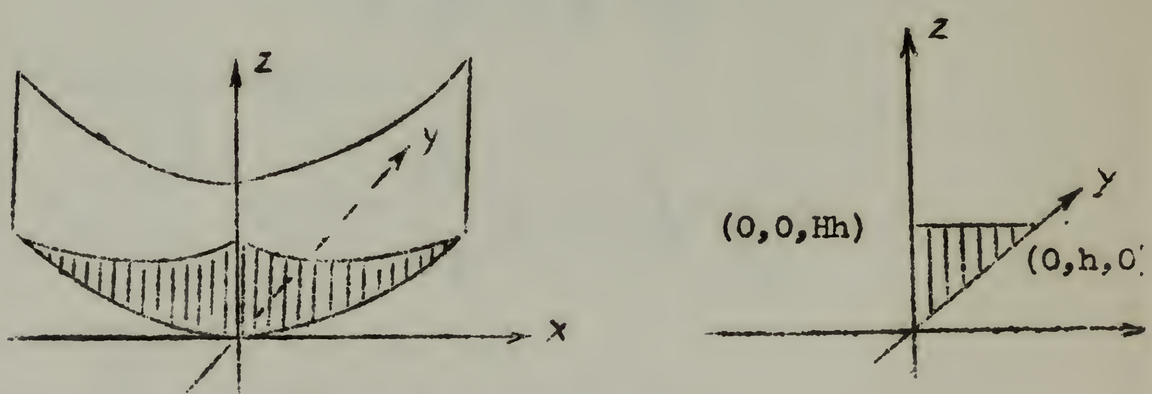


Fig. 9

vertical reactions, a plane is passed parallel to the Y-axis and through the points $(\frac{L}{2},0,0)$ and $(w,0,\frac{ab}{L})$. Another plane is passed parallel to the Y-axis and through the points $(-\frac{L}{2},0,0)$ and $(w,0,\frac{ab}{L})$. This sketch is shown in Fig. 7. The area to be found is that formed by the intersection of the parabolic cylinder, the two planes given, and the YZ-plane. The distance from the center-line of the arch to the position of the unit load under consideration is "w", and the distances from each end of the span to the unit load are "a" and "b" respectively. Hence, $a = \frac{L}{2} - w$ and $b = \frac{L}{2} + w$.

Since both "E" and "I" are constant in this arch, and they appear in the same way in every $\frac{M}{EI}$ diagram, for the sake of simplicity they will be dropped in future calculations.

The equation of plane 1 is:

$$z_1 = \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right)$$

The equation of plane 2 is:

$$z_2 = \left(\frac{L}{2} - w\right)\left(\frac{1}{2} + \frac{x}{L}\right)$$

To find the area:

$$A_1 = \int_w^{\frac{L}{2}} z_1 \, ds = \int_w^{\frac{L}{2}} \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right) ds$$

$$A_1 = \int_w^{\frac{L}{2}} \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right)\frac{2}{K}\sqrt{\frac{K^2}{4} + x^2} \; dx$$

where ds = increment of arc length = $\sqrt{dx^2 + dy^2}$

$$ds = \sqrt{1 + \left(\frac{2x}{K}\right)^2}\, dx = \frac{2}{K}\sqrt{\frac{K^2}{4} + x^2}$$

Then, $$A_1 = \left(\frac{L}{2} + w\right)\frac{2}{K}\frac{1}{2}\left[\frac{x}{2}\sqrt{\frac{K^2}{4} + x^2} + \frac{K^2}{8}\ln\left(x + \sqrt{\frac{K^2}{4} + x^2}\right)\right]$$

$$-\left(\frac{L}{2} + w\right)\frac{2}{K}\;\frac{1}{L}\frac{1}{3}\left(\frac{K^2}{4} + x^2\right)^{\frac{3}{2}}$$

$$A_1 = \left(\frac{L}{2} + w\right)\frac{1}{K}\left[\frac{L}{4}\sqrt{\frac{K^2}{4} + \frac{L^2}{4}} - \frac{w}{2}\sqrt{\frac{K^2}{4} + w^2} + \frac{K^2}{8}\ln\;\frac{L}{2} + \sqrt{\frac{K^2}{4} + \frac{L^2}{4}}\right.$$

$$\left. - \frac{K^2}{8}\ln\left(w + \sqrt{\frac{K^2}{4} + w^2}\right) - \left(\frac{L}{2} + w\right)\frac{2}{3KL}\left[\left(\frac{K^2}{4} + \frac{L^2}{4}\right)^{\frac{3}{2}} - \left(\frac{K^2}{4} + w^2\right)^{\frac{3}{2}}\right]\right]$$

For simplicity, let $R = \sqrt{L^2 + K^2}$ and $N = \sqrt{\frac{K^2}{4} + w^2}$

Then, $$A_1 = \left(\frac{L}{2} + w\right)\frac{1}{K}\left[\frac{LR}{8} - \frac{wN}{2} - \frac{R^3}{12L} + \frac{2N^3}{3L} - \frac{K^2}{8}\ln 2\left(\frac{w + N}{R + L}\right)\right]$$

Similarly $$A_2 = \int_{-\frac{L}{2}}^{w} z_2\, ds = \int_{-\frac{L}{2}}^{w}\left(\frac{L}{2} - w\right)\left(\frac{1}{2} + \frac{x}{L}\right)\frac{2}{K}\sqrt{\frac{K^2}{4} + x^2}\, dx$$

Integrating, substituting limits, and simplifying, the result is:

$$A_2 = \left(\frac{L}{2} - w\right)\frac{1}{K}\left[\frac{LR}{8} + \frac{wN}{2} - \frac{R^3}{12L} + \frac{2N^3}{3L} - \frac{K^2}{8}\ln\frac{1}{2}\left(\frac{R - L}{w + N}\right)\right]$$

$$A_p = A_1 + A_2 = \frac{L}{2K}\left[\frac{LR}{4} + \frac{4N^3}{3L} - \frac{R^3}{6L} + \frac{K^2}{8}\ln\left(\frac{R + L}{R - L}\right)\right]$$

$$- \frac{w}{K}\left[wN + \frac{K^2}{8}\ln 4\left(\frac{w + N}{K}\right)^2\right] \tag{1}$$

The next area to be found is that of the $\frac{M}{EI}$ diagram for the redundant fixed end moment on the right end, as shown in Fig. 4. Again the calculus is employed, with the same parabolic cylinder placed as was done in finding the area for the unit load. A plane is again passed parallel to the Y-axis, but through the points $(\frac{L}{2},0,M_R)$ and $(-\frac{L}{2},0,0)$ as shown in Fig. 8.

The equation of the plane is:

$$z = \frac{M_R}{2} + x\frac{M_R}{L} = M_R\left(\frac{1}{2} + \frac{x}{L}\right)$$

$$A_{M_R} = \int_{-\frac{L}{2}}^{\frac{L}{2}} z\,ds = \int_{-\frac{L}{2}}^{\frac{L}{2}} M_R\left(\frac{1}{2} + \frac{x}{L}\right)\frac{2}{K}\sqrt{\frac{K^2}{4} + x^2}\,dx$$

$$A_{M_R} = \frac{M_R}{K}\left[\frac{x}{2}\sqrt{\frac{K^2}{4} + x^2} + \frac{K^2}{8}\ln\left(x + \sqrt{\frac{K^2}{4} + x^2}\right)\right] + \frac{2M_R}{3LK}\left(\frac{K^2}{4} + x^2\right)^{\frac{3}{2}}$$

$$= \frac{M_R}{K}\left[\frac{LR}{4} + \frac{K^2}{8}\ln\left(\frac{L+R}{2}\right) - \frac{K^2}{8}\ln\left(\frac{R-L}{2}\right)\right] + \frac{2M_R}{3LK}\left(\frac{R^3}{8} - \frac{R^3}{8}\right)$$

$$A_{M_R} = \frac{M_R}{4}\left[\frac{LR}{K} + \frac{K}{2}\ln\left(\frac{R+L}{R-L}\right)\right] \qquad (2)$$

Similarly, the area of the $\frac{M}{EI}$ diagram for the fixed end moment at the left end is:

$$A_{M_L} = \frac{M_L}{4}\left[\frac{LR}{K} + \frac{K}{2}\ln\left(\frac{R+L}{R-L}\right)\right] \qquad (3)$$

The last area to be found is that of the $\frac{M}{EI}$ diagram for the horizontal thrust. Again the same parabolic cylinder is used, but this time a plane is passed parallel

to the X-axis and through the points (0,0,Hh) and (0,h,0) as shown in Fig. 9.

The equation of the plane is:

$$z = hH - Hy = hH - H\frac{x^2}{K}$$

$$A_H = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(hH - \frac{H}{K}x^2\right)\frac{2}{K}\sqrt{\frac{K^2}{4} + x^2}\ dx$$

$$= \frac{2hH}{K}\left[\frac{x}{2}\sqrt{\frac{K^2}{4} + x^2} + \frac{K^2}{8}\ln\left(x + \sqrt{\frac{K^2}{4} + x^2}\right)\right]$$

$$- \frac{2H}{K^2}\left[\frac{x}{4}\left(\frac{K^2}{4} + x^2\right)^{\frac{3}{2}} - \frac{K^2}{16}\left(\frac{x}{2}\sqrt{\frac{K^2}{4} + x^2} + \frac{K^2}{8}\ln\left(x + \sqrt{\frac{K^2}{4} + x^2}\right)\right)\right]$$

$$= \frac{2hH}{K}\left[\frac{LR}{4} + \frac{K^2}{8}\ln\left(\frac{R + L}{R - L}\right)\right] - \frac{2H}{K^2}\left[\frac{LR^3}{32} - \frac{K^2}{16}\left(\frac{LR}{4} + \frac{K^2}{8}\ln\left[\frac{R + L}{R - L}\right]\right)\right]$$

$$A_H = \frac{H}{8}\left(\left[\frac{1}{4} + \frac{4h}{K}\right]\left[LR + \frac{K^2}{2}\ln\left(\frac{R + L}{R - L}\right)\right] - \frac{LR^3}{2K^2}\right) \tag{4}$$

The areas of all the $\frac{M}{EI}$ diagrams have now been found. The second step is to find the centroid distances of these areas.

UNIT LOAD. For the $\frac{M}{EI}$ diagram of the unit load, let "d" equal the distance along the X-axis from the centerline of the arch to the centroid of the area shown in Fig. 7.

Then $d = \bar{x} = \frac{M_p'}{A_p}$ where M_p' = first moment about centerline.

$$M_p' = \int_{-\frac{L}{2}}^{\frac{L}{2}} xz\ ds = \int_{w}^{\frac{L}{2}} x\left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right)\frac{2}{K}\sqrt{\frac{K^2}{4} + x^2}\ dx$$

$$+ \int_{-\frac{L}{2}}^{w} x\left(\frac{L}{2} - w\right)\left(\frac{1}{2} + \frac{x}{L}\right)\frac{2}{K}\sqrt{\frac{K^2}{4} + x^2}\ dx$$

$$M_p' = \left(\frac{L}{2} + w\right)\frac{1}{K}\frac{1}{3}\left(\frac{K^2}{4} + x^2\right)^{\frac{3}{2}} - \left(\frac{L}{2} + w\right)\frac{2}{LK}\left[\frac{x}{4}\sqrt{\frac{K^2}{4} + x^2}^{\,3}\right.$$

$$\left. - \frac{K^2}{16}\left(\frac{x}{2}\sqrt{\frac{K^2}{4} + x^2} + \frac{K^2}{8}\ln\left[x + \sqrt{\frac{K^2}{4} + x^2}\right]\right)\right] \text{from } w \text{ to } \frac{L}{2}$$

[illegible] through the points [illegible] and [illegible] as shown in Fig. 7.

The equation of the curve is

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

[illegible]

[illegible] shown in Fig. 7. [illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$+\left(\frac{L}{2} - w\right) \frac{1}{K} \frac{1}{3} \left(\frac{K^2}{4} + x^2\right)^{\frac{3}{2}} + \left(\frac{L}{2} - w\right) \frac{2}{LK} \left[\frac{x}{4} \left(\frac{K^2}{4} + x^2\right)^{\frac{3}{2}}\right.$$

$$\left. - \frac{K^2}{16} \left(\frac{x}{2}\sqrt{\frac{K^2}{4} + x^2} + \frac{K^2}{8} \ln\left(x + \sqrt{\frac{K^2}{4} + x^2}\right)\right)\right] \text{from } -\frac{L}{2} \text{ to } w.$$

$$M'_p = \left(\frac{L}{2} + w\right) \frac{1}{3K} \left(\frac{R^3}{8} - N^3\right) - \left(\frac{L}{2} + w\right) \frac{2}{LK} \left[\frac{LR^3}{64} - \frac{wN^3}{4}\right.$$

$$\left. - \frac{K^2}{16} \left(\frac{LR}{8} - \frac{wN}{2} + \frac{K^2}{8} \ln \frac{1}{2}\left[\frac{R + L}{w + N}\right]\right)\right] + \left(\frac{L}{2} - w\right) \frac{1}{3K} \left(N^3 - \frac{R^3}{8}\right)$$

$$+ \left(\frac{L}{2} - w\right) \frac{2}{LK} \left[\frac{wN^3}{4} + \frac{LR^3}{64} - \frac{K^2}{16}\left(\frac{wN}{2} + \frac{LR}{8} + \frac{K^2}{8} \ln 2\left(\frac{w + N}{R - L}\right)\right)\right]$$

$$M'_p = \frac{wR^3}{48K} - \frac{wN^3}{6K} - \frac{wNK}{16} + \frac{wRK}{132} - \frac{K^3}{128}\ln 4\left(\frac{w + N}{K}\right)^2 + \frac{wK^3}{64L} \ln\left(\frac{R + L}{R - L}\right) \tag{5}$$

$$d \doteq \bar{x} = \frac{M'_p}{A_p} = \frac{\text{Equation (5)}}{\text{Equation (1)}} \tag{5a}$$

To find "r" $= \bar{y}$ of the centroid for the unit load diagram,

$$M''_p = \int_{-\frac{L}{2}}^{\frac{L}{2}} yz \, ds$$

$$M''_p = \int_{w}^{\frac{L}{2}} \frac{x^2}{K} \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right)\frac{2}{K}\sqrt{\frac{K^2}{4} + x^2} \, dx$$

$$+ \int_{-\frac{L}{2}}^{w} \frac{x^2}{K}\left(\frac{L}{2} - w\right)\left(\frac{1}{2} + \frac{x}{L}\right) \frac{2}{K} \sqrt{\frac{K^2}{4} + x^2} \, dx$$

Integrating, substituting limits, simplifying, and making the substitutions of N and R as above, the following result is obtained:

$$M''_p = \frac{L^2R^3}{320K^2} - \frac{w^2N^3}{10K^2} - \frac{L^2R}{128} + \frac{w^2N}{16} + \frac{R^3}{120} - \frac{N^3}{15} - \frac{K^2L}{256} \ln\left(\frac{R + L}{R - L}\right)$$

$$+ \frac{wK^2}{128} \ln 4\left(\frac{w + N}{K}\right)^2 \tag{6}$$

$$r = \bar{y} = \frac{M''_p}{A_p} = \frac{\text{Equation (6)}}{\text{Equation (1)}} \tag{6a}$$

MOMENT AT RIGHT SPRINGING. To find the centroid of the $\frac{M}{EI}$ diagram for the springing moment at the right end of the arch, the same procedure is used as that employed for the centroid distances of the unit load $\frac{M}{EI}$ diagram.

For "c" = $\bar{x}$:

$$M'_{M_R} = \int_{-\frac{L}{2}}^{\frac{L}{2}} xz\ ds = \int_{-\frac{L}{2}}^{\frac{L}{2}} x\ M_R \left(\frac{1}{2} + \frac{x}{L}\right) \frac{2}{K} \sqrt{\frac{K^2}{4} + x^2}\ dx$$

Integrating, substituting limits, simplifying, and making the substitutions of N and R as before, the following result is obtained:

$$M'_{M_R} = M_R \left[\frac{R^3}{16K} - \frac{RK}{32} - \frac{K^3 \ln\left(\frac{R+L}{R-L}\right)}{64L}\right] \tag{7}$$

$$c = \bar{x} = \frac{\text{Equation (7)}}{\text{Equation (2)}} = \frac{R^3}{4LR + 2K^2 \ln\left(\frac{R+L}{R-L}\right)} - \frac{K^2}{8L} \tag{8}$$

For "m" = $\bar{y}$: $M''_{M_R} = \int_{-\frac{L}{2}}^{\frac{L}{2}} yz\ ds$

$$M''_{M_R} = M_R \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{x^2}{K}\left(\frac{1}{2} + \frac{x}{L}\right) \frac{2}{K} \sqrt{\frac{K^2}{4} + x^2}\ dx$$

Integrating, substituting limits, and simplifying, the following result is obtained:

$$M''_{M_R} = \frac{M_R}{8}\left[\frac{LR^3}{42K^2} - \frac{LR}{8} - \frac{K^2}{16} \ln \frac{R+L}{R-L} + \frac{2R^3}{15L}\right] \tag{9}$$

$$m = \bar{y} = \frac{\text{Equation (9)}}{\text{Equation (2)}} = \frac{L}{2K} \frac{R^3}{4LR + 2K^2 \ln \frac{R+L}{R-L}} - \frac{K}{16} = \frac{L}{2K}\, c \tag{10}$$

MOMENT AT LEFT SPRINGING. In a similar manner, neglecting the sign difference for $\bar{x}$, the equations for the centroid distances of the $\frac{M}{EI}$ diagram for the springing moment at the left end are the same as those for the right end.

moment of inertia increases, to find the centroid of the $\frac{M}{EI}$ diagram for the parabolic moment in the first half of the beam, the same procedure is used as that developed for the remaining distance of the beam from $\frac{M}{EI}$ diagrams.

For $I/I' = 2$

$$[illegible]$$

Integrating, substituting limits, simplifying, and making the substitution of a and b as before, the following simple expression:

$$[illegible]$$

$$[illegible]$$

For $I/I' = 3$

$$[illegible]$$

$$[illegible]$$

Integrating, substituting limits, and simplifying, the following result is obtained:

$$[illegible]$$

$$[illegible]$$

moment of inertia variation. In a similar manner, neglecting the minor difference for I_2, the equations for the centroid distance of the $\frac{M}{EI}$ diagram for the parabolic moment of the left end are the same as those for the right end.

HORIZONTAL THRUST. To find the centroid distances of the $\frac{M}{EI}$ diagram for the horizontal thrust, the same procedure is followed as that in the previous derivations. Since the thrust diagram is symmetrical with respect to the Y-axis, the value of $\bar{x}$ equals 0. To find $\bar{y}$:

$$n = \bar{y} = \frac{M_H''}{A_H} \qquad M_H'' = \int_{-\frac{L}{2}}^{\frac{L}{2}} yz\ ds = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{x^2 H}{K}\left(h - \frac{x^2}{K}\right)\frac{2}{K}\sqrt{\frac{K^2}{4} + x^2}\ dx$$

Integrating, substituting limits, simplifying, and making the proper substitutions of N and R, the following result is obtained:

$$M_H'' = \frac{H}{K}\left[\left(\frac{1}{2} + \frac{4h}{K}\right)\left(\frac{LR^3}{64} - \frac{LRK^2}{128} - \frac{K^4}{256}\ \ln\left[\frac{R + L}{R - L}\right]\right) - \frac{L^3R^3}{96K^2}\right] \tag{13}$$

$$n = \frac{M_H''}{A_H} = \frac{\text{Equation (13)}}{\text{Equation (4)}} \tag{14}$$

For the sake of convenience a list of the centroid distance designations is presented here.

Letter designation	Quantity	Equation No.
d	$\bar{x}_p$	(5a)
r	$\bar{y}_p$	(6a)
c	$\bar{x}_{M_R}$ and $\bar{x}_{M_L}$	(8)
m	$\bar{y}_{M_R}$ and $\bar{y}_{M_L}$	(10)
n	$\bar{y}_H$	(14)

Having obtained all the areas of the $\frac{M}{EI}$ diagrams, and expressions for the centroid distances of those areas, the third step is to place the diagrams on the conjugate structure, set up the equilibrium equations for this

conjugate structure, and solve these equations. Fig. 10 shows a plan view of the conjugate structure used when the unit load is placed at the center or to the left of center. Here, the centroids of the $\frac{M}{EI}$ diagrams have been shown, with the areas themselves considered concentrated at those points.

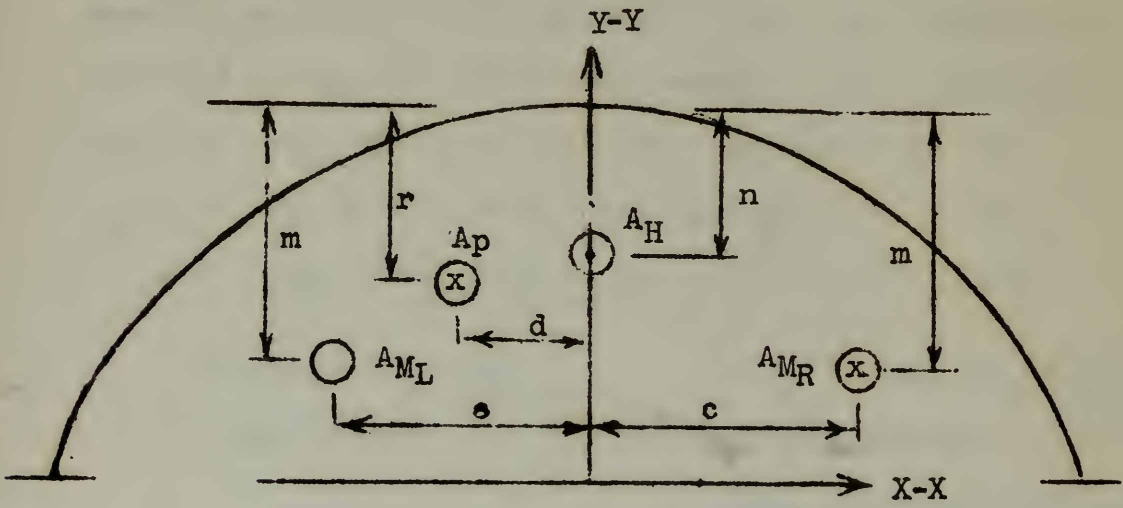


Fig. 10

In this case both the diagram for the unit load and that for the right springing cause tension on the inside of the arch and will act downward. The diagram for the horizontal thrust will act upward, since that force always tends to cause compression on the inside.

An "X" indicates that the load acts downward, and a dot indicates that the load acts upward. The centroid for the diagram due to the moment at the left springing is left undesignated, because it is not known in just what direction it will act. Both its magnitude and direction vary with the position of the unit load. However, this diagram will be assumed to act downward; then if the

result comes out positive, the ~~right~~ left springing moment is positive (tends to cause tension on the inside of the real frame). If the result comes out negative, the moment is negative (tends to cause compression on the inside of the real frame).

Now the equilibrium equations can be set up. Taking moments parallel to the XX and YY axes, and through the centroid of A_{M_L} the following equations are obtained:

$M_{XX} = 0 \quad A_{M_R}(2c) + A_p(c - d) - A_H(c) = 0. \quad (a)$

$M_{YY} = 0 \quad A_p(m - r) - A_H(m-n) = 0. \quad (b)$

$V = 0 \quad A_{M_R} + A_{M_L} + A_p - A_H = 0. \quad (c)$

Transposing (b)

$$A_H = A_p\left(\frac{m-r}{m-n}\right) \qquad (15)$$

Substituting (15) in (a) and transposing:

$$A_{M_R} = \frac{A_p}{2}\left(\frac{m-r}{m-n} + \frac{d}{c} - 1\right) \qquad (16)$$

Sub. (15) and (16) in (c) and simplifying:

$$A_{M_L} = \frac{A_p}{2}\left(\frac{m-r}{m-n} - \frac{d}{c} - 1\right) \qquad (17)$$

Equations (15), (16), and (17) are the expressions for the values of the unknown moments and the thrust. At this point in the use of the method, everything on the right hand side of the equations is known, and the expressions on the left hand side each contain one of the unknown values. These equations have been developed for a unit load at center or to the left of center, so they are VALID ONLY WHEN THE LOAD IS PLACED TO THE LEFT OF CENTER OR AT THE CENTER. When the unit load is placed to

the right of center, the right hand side of the equations remains unchanged, but A_{M_R} and A_{M_L} must be INTERCHANGED IN EQUATIONS (16) and (17).

An example will be given to illustrate the procedure to be followed in the solution of the parabolic arch; using a 10 foot span and a 2 foot rise, with the unit load placed one foot to the left of center of the span, the procedure is as follows:

$$L = 10' \qquad h = 2' \qquad w = 1'$$

$$K = \frac{L^2}{4h} = \frac{100}{4x2} = 12.5$$

$$R = \sqrt{K^2 + L^2} = \sqrt{156.25 + 100} = 16.0078$$

$$N = \sqrt{\frac{K^2}{4} + w^2} = \sqrt{\frac{156.25}{4} + 1} = 6.3295$$

$$\ln\left(\frac{R + L}{R - L}\right) = \ln\left(\frac{16.0078 + 10}{16.0078 - 10}\right) = 1.4653$$

$$\ln 4\left(\frac{w + N}{K}\right)^2 = \ln 4\left(\frac{1 + 6.3295}{12.5}\right)^2 = 0.3177$$

$$A_p = \frac{L}{2K}\left[\frac{LR}{4} + \frac{4N^3}{3L} - \frac{R^3}{6L} + \frac{K^2}{8}\ln\left(\frac{R + L}{R - L}\right)\right] - \frac{w}{K}\left[wN + \frac{K^2}{8}\ln 4\left(\frac{w + N}{K}\right)^2\right]$$

$$A_p = \frac{10}{2x12.5}\left[\frac{10x16.0078}{4} + \frac{4x(6.3295)^3}{3x10} - \frac{(16.0078)^3}{6x10} + \frac{156.25}{8}(1.4653)\right] - \frac{1}{12.5}\left[1x6.3295 + \frac{156.25}{8}(0.3177)\right]$$

$$A_p = 12.6050$$

$$A_{M_R} = \frac{M_R}{4}\left[\frac{LR}{K} + \frac{K}{2}\ln\left(\frac{R + L}{R - L}\right)\right]$$

$$A_{M_R} = \frac{M_R}{4}\left[\frac{10\text{x}16.0078}{12.5} + \frac{12.5}{2}(1.4653)\right] = 5.491\ M_R$$

Since the numerical value of A_{M_L} = the numerical value of A_{M_R}, $A_{M_L} = 5.491\ M_L$

$$A_H = \frac{H}{8}\left(\left[\frac{1}{4} + \frac{4h}{K}\right]\left[LR + \frac{K^2}{2}\ \ln\left(\frac{R + L}{R - L}\right)\right] - \frac{LR^3}{2K^2}\right) \qquad (4)$$

$$A_H = \frac{H}{8}\left(\left[\frac{1}{4} + \frac{4\text{x}2}{12.5}\right]\left[10\text{x}16.0078 + \frac{156.25}{2}(1.4653)\right] - \frac{10\text{x}16.0078^3}{2\text{x}156.25}\right)$$

$A_H = 14.137\ H$

To find the centroid distances of these four areas, equations (5) to (14) will be used.

$$c = \frac{R^3}{4LR \quad 2K^2\ \ln\left(\frac{R + L}{R - L}\right)} - \frac{K^2}{8L} \qquad (8)$$

$$c = \frac{(16.0078)^3}{4\text{x}10\text{x}16.0078 + 2\text{x}156.25(1.4653)} - \frac{156.25}{8\text{x}10} = 1.782$$

$$d = \frac{1}{A_R}\left(\frac{wR^3}{48K} - \frac{wN^3}{6K} - \frac{wNK}{16} + \frac{wRK}{32} - \frac{K^3}{128}\ \ln\ 4\left(\frac{w + N}{K}\right)^2 + \frac{wK^3}{64L}\ln\left(\frac{R + L}{R - L}\right)\right) \quad (5.)$$

$$d = \frac{1}{12.605}\left[\frac{1\text{x}16.0078^3}{48\text{x}12.5} - \frac{1\text{x}6.3295^3}{6\text{x}12.5} - \frac{1\text{x}6.3295\text{x}12.5}{16} + \frac{1\text{x}16.0078\text{x}12.5}{32} - \frac{12.5^3}{128}(0.3177) + \frac{1\text{x}12.5^3(1.4653)}{64\text{x}10}\right]$$

$$d = \frac{4.381}{12.605} = 0.3475$$

$$m = \left(\frac{L}{2K}\right)c = \frac{10\text{x}1.782}{25} = 0.7128 \qquad (10)$$

$$n = \frac{1}{A_H}\left\{\frac{H}{K}\left[\left(\frac{1}{2} + \frac{4h}{K}\right)\left(\frac{LR^3}{64} - \frac{LRK^2}{128} - \frac{K^4}{256}\ \ln\left(\frac{R + L}{R - L}\right)\right) - \frac{L^3R^3}{96K^2}\right]\right\} \qquad (14)$$

$$n = \frac{1}{14.137H}\left\{\frac{H}{12.5}\left[\left(\frac{1}{2}+\frac{4x2}{12.5}\right)\left(\frac{10x16.0078^3}{64}\;\frac{-10x16.0078x156.2}{128}\right.\right.\right.$$
$$\left.\left.\left.-\frac{12.5^4}{256}(1.4653)\right)-\frac{10^3 \ x\ 16.0078^3}{96x(12.5)^2}\right]\right\}$$

$$n = \frac{75.11}{14.137x12.5} = 0.4250$$

$$r = \frac{1}{A_p}\left[\frac{L^2R^3}{320K^2} - \frac{w^2N^3}{10K^2} - \frac{L^2R}{128} + \frac{w^2N}{16} + \frac{R^3}{120} - \frac{N^3}{15}\right.$$
$$\left. - \frac{K^2L}{256}\ \ln\left(\frac{R+L}{R-L}\right) + \frac{wK^2}{128}\ \ln\left(\frac{w+N}{K}\right)^2\right] \quad (6)$$

$$r = \frac{1}{12.605}\left[\frac{10^2x16.0078^3}{320x156.25} - \frac{1^2x6.3295^3}{10x156.25} - \frac{10^2x16.0078}{128}\right.$$
$$+ \frac{1^2x6.3295}{16} + \frac{16.0078^3}{120} - \frac{6.3295^3}{15} - \frac{156.25x10}{256}(1.4653)$$
$$\left. + \frac{1x156.25}{128}(0.3177)\right]$$

$$r = \frac{4.6533}{12.605} = 0.3692$$

Knowing the values of the various areas and centroid distances, the last step is to substitute these values in equations (15), (16), and (17).

$$A_H = A_p\left(\frac{m-r}{m-n}\right) \quad 14.137H = 12.605\left(\frac{.7128-.3692}{.7128-.4250}\right) \quad (15)$$

$$H = \frac{12.605}{14.137}(1.1938) = 1.064 \qquad H = 1.064$$

$$A_{M_R} = \frac{A_p}{2}\left(\frac{m-r}{m-n} + \frac{d}{c} - 1\right) \quad (16)$$

$$5.491\ M_R = \frac{12.605}{2}\left(1.1938 + \frac{.3475}{1.782} - 1\right)$$

$$M_R = 0.4467$$

$$A_{M_L} = \frac{A_p}{2}\left(\frac{m-r}{m-n} - \frac{d}{c} - 1\right) \tag{17}$$

$$5.491\ M_L = \frac{12.605}{2}\left(1.1938 - \frac{.3475}{1.782} - 1\right)$$

$$M_L = -0.0011$$

Here it is seen that M_L is negative, showing that this moment tends to cause compression on the inside of the real structure.

When the unit load is placed one foot to the right of center, "H" is the same as for the example above, but the values of A_{M_R} and AM_L must be interchanged, so that for the unit load in this new position, the corresponding values are:

$H = 1.064$ $\qquad M_R = -0.0011$ $\qquad M_L = 0.4467$

To find the values of H, M_R, and M_L for the remaining points on the span, the same procedure is followed, substituting the proper value of "w" in each case. However, for each new load point only one of the areas and two of the centroid distances must be recomputed, in addition to the three final equations.

If it be desired to find the influence lines by using ten points, as is often done, the above procedure is required for only five of the points, since the values for the symmetrical points can be found by interchanging M_R and M_L as was shown in the example.

This, then, completes the derivation and application of the general method of solution for parabolic arches whose cross-sectional moment of inertia is constant.

RELATIONSHIPS BETWEEN ARCHES HAVING THE SAME RATIO OF RISE TO SPAN

For arches having the same ratio of rise to span, certain relationships can be developed which will save a great deal of time in the solution of a series of arches. Using the same equations as in PART I of this thesis, and keeping the ratio of rise to span constant, the following simplifications can be made:

Let $k = \frac{h}{L}$, the rise to span ratio, in this case constant.

Let $v = \frac{w}{L}$, where "v" is a constant indicating the load point; i.e., 0.1 the span off center, 0.2 off center, etc.

Then $K = \frac{L^2}{4h} = \frac{L}{4k}$

$$R = \sqrt{L^2 + K^2} = \sqrt{L^2 + \frac{L^2}{16k^2}} = L\sqrt{1 + \frac{1}{16k^2}} = LC_1$$

$$N = \sqrt{w^2 + \frac{K^2}{4}} = \sqrt{v^2L^2 + \frac{L^2}{64k^2}} = L\sqrt{v^2 + \frac{1}{64k^2}} = LC_2$$

$$\ln\left(\frac{R + L}{R - L}\right) = \ln \frac{L\sqrt{1 + \frac{1}{16k^2}} + L}{L\sqrt{1 + \frac{1}{16k^2}} - L} = \ln \frac{\sqrt{1 + \frac{1}{16k^2}} + 1}{\sqrt{1 + \frac{1}{16k^2}} - 1} = C_3$$

$$\ln 4\left(\frac{w + N}{K}\right)^2 = \ln 4\left(\frac{vL + \sqrt{(vL)^2 + \frac{L^2}{64k^2}}}{\frac{L}{4k}}\right)^2$$

$$= \ln 4\left(\frac{v \quad \sqrt{v^2 + \frac{1}{64k^2}}}{\frac{1}{4k}}\right)^2 = C_4$$

The values of the constants C_1 to C_4 can be computed if desired, but the only point of interest to be noted here is that they are constant values for the same load point on arches having the same ratio of rise to span. All constants to be used in the following discussion will simply be noted as C_5 , C_6 , etc.

Substituting the above values in the equations of PART I, the following simplifications are made:

In Eq'n. (1) $$\underline{A_p} = \frac{L}{\frac{2L}{4k}}\left[\frac{LxLC_1}{4} + \frac{4L^3C_2^3}{3L} - \frac{L^3C_1^3}{6L} + \frac{L^2C_3}{128k^2}\right] - \frac{vL}{\frac{L}{4k}}\left[vLxLC_2 + \frac{L^2C_4}{128k^2}\right] = L^2C_5$$

In Eq'n. (2) $$A_{M_R} = \frac{M_R}{4}\left[\frac{LxLC_1}{\frac{L}{4k}} + \frac{LC_3}{8k}\right] = LC_6M_R$$

In Eq'n. (3) by a similar simplification, $A_{M_L} = LC_6M_L$

In Eq'n. (4) $$A_H = \frac{H}{8}\left\{\left[\frac{1}{4} + \frac{4h}{\frac{L^2}{4h}}\right]\left[LxLC_1 + \frac{L^2C_3}{32k^2}\right] - \frac{LxL^3C_1^3}{\frac{2L^2}{16k^2}}\right\} = HL^2C_7$$

For the equations having to do with the centroid distances:

$$c = \frac{L^3C_1^3}{4LxLC_1 + \frac{2L^2C_3}{16k^2}} - \frac{L^2}{128k^2L} = LC_8$$

$$d = \frac{1}{L^2C_5}\left[\frac{vLxL^3C_1^3}{48\frac{L}{4k}} - \frac{vLxL^3C_2^3}{6\frac{L}{4k}} - \frac{vLxLC_2L}{64k} + \frac{vLxLC_1L}{128k} - \frac{L^3C_4}{128(4k)^4} + \frac{vLxL^3C_3}{64L(4k)^3}\right] = LC_9$$

$$m = \frac{L}{2K}\,c = \frac{L}{\frac{2L}{4k}} \times LC_8 = LC_{10}$$

$$n = \frac{1}{HL^2C_7}\left\{ \frac{H}{\frac{L}{4k}}\left[\left(\frac{1}{2} + \frac{4h}{\frac{L^2}{4h}}\right)\left(\frac{LxL^3C_1^3}{64} - \frac{LxLC_1L^2}{128(4k)^2} - \frac{L^4C_3}{256(4k)^4}\right) - \frac{L^3L^3C_1^3}{\frac{96L^2}{16k^2}}\right]\right\} = LC_{11}$$

$$r = \frac{1}{L^2C_5}\left[\frac{L^2L^3C_1^3}{\frac{320L^2}{16k^2}} - \frac{v^2L^2L^3C_2^3}{\frac{10L^2}{16k^2}} - \frac{L^2LC_1}{128} + \frac{v^2L^2LC_2}{16} + \frac{L^3C_1^3}{120} - \frac{L^3C_2^3}{15} - \frac{L^2LC_3}{256(4k)^2} + \frac{vLxL^2C_4}{128(4k)^2}\right] = LC_{12}$$

Now, applying the above simplifications in the final equations, the following results are obtained:

$$A_H = A_p\left(\frac{m-r}{m-n}\right) \qquad HL^2C_7 = L^2C_5\left(\frac{LC_{10} - LC_{12}}{LC_{10} - LC_{11}}\right) \qquad H = C_{13} \tag{18}$$

$$A_{M_R} = \frac{A_p}{2}\left[\frac{m-r}{m-n} + \frac{d}{c} - 1\right] \qquad LC_6M_R = \frac{L^2C_5}{2}\left(\frac{LC_{10} - LC_{12}}{LC_{10} - LC_{11}} + \frac{LC_9}{LC_8} - 1\right)$$

$$M_R = LC_{14} \tag{19}$$

$$A_{M_L} = \frac{A_p}{2}\left[\left(\frac{m-r}{m-n}\right) - \frac{d}{c} - 1\right] \qquad LC_6M_L = \frac{L^2C_5}{2}\left(\frac{LC_{10} - LC_{12}}{LC_{10} - LC_{11}} - \frac{LC_9}{LC_8} - 1\right)$$

$$M_L = LC_{15} \tag{20}$$

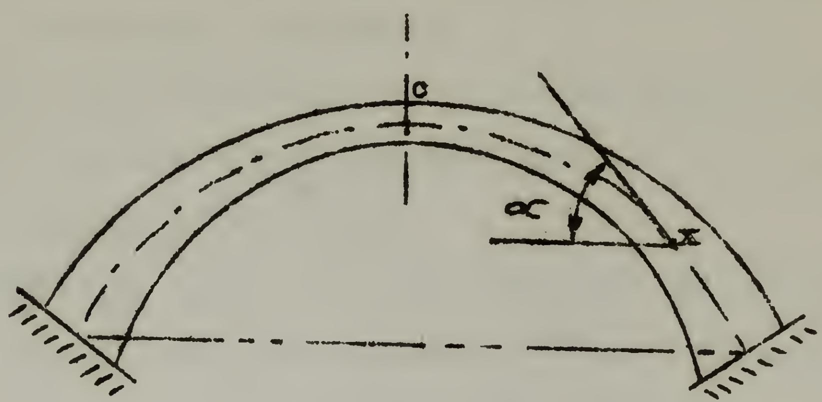
The last three equations bring out the important

conclusions that for the same load point on arches having the same ratio of rise to span, (1) THE VALUE OF THE HORIZONTAL THRUST IS CONSTANT, AND (2) THE VALUE OF THE SPRINGING MOMENTS VARIES DIRECTLY AS THE LENGTH OF SPAN.

For example, if the moment at the right springing on an arch 10' in span and having a ratio of rise to span of 0.20 is 0.447, then the moment at the right springing on an arch with a span of 20' and ratio of rise to span of still 0.20 is 2x0.447 OR 0.894, providing the same load point is considered.

PART II : VARYING MOMENT OF INERTIA

In this section the same principles and general procedure are used as those found in PART I. However, in this case, the arch has a cross-sectional moment of inertia that varies according to a specific function. Fig. 11 shows that the moment of inertia of the cross-section of this arch at any point is equal to the moment of inertia at the center times the secant of the angle "alpha".



$$I_x = I_c \sec \alpha$$

Fig. 11

Again the modulus of elasticity, E, will be dropped in calculations, since it appears in the same way in all equations, and consequently cancels out. However, the moment of inertia must be carried along because it is a varying quantity.

To derive the expression for the moment of inertia at any point the procedure is as follows:

$$I_x = I_c \sec \alpha = I_c \frac{ds}{dx}$$

The ordinates of the $\frac{M}{EI}$ diagrams are exactly the same as those for the arch with the constant moment of inertia in PART I, Hence the equations of all the planes that were passed through the parabolic cylinders in the development of PART I remain unchanged. The only difference from the development in PART I is that in the development to be shown now, the moment of inertia will appear in all of the preliminary equations.

The first step again is to derive equations for the areas of the various $\frac{M}{EI}$ diagrams that are to be applied to the conjugate structure.

Referring to Fig. 7, and using the same terminology that appears there, the equations of the two planes are:

$$z_1 = \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right)\frac{1}{I_x}$$

$$z_2 = \left(\frac{L}{2} - w\right)\left(\frac{1}{2} + \frac{x}{L}\right)\frac{1}{I_x}$$

$$A_1 = \int_w^{\frac{L}{2}} z_1 \, ds = \int_w^{\frac{L}{2}} \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right)\frac{1}{I_x} \, ds$$

$$A_1 = \int_w^{\frac{L}{2}} \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right)\frac{1}{I_c\frac{ds}{dx}} \, ds = \frac{1}{I_c}\int_w^{\frac{L}{2}} \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right) dx$$

$$A_1 = \frac{1}{I_c}\left(\frac{L}{2} + w\right)\left(\frac{x}{2} - \frac{x^2}{2L}\right) = \frac{1}{I_c}\left(\frac{L}{2} + w\right)\left(\frac{L}{4} - \frac{w}{2} - \frac{L^2}{8L} + \frac{w^2}{2L}\right)$$

$$A_1 = \frac{1}{I_c}\left(\frac{L}{2} + w\right)\left(\frac{L}{8} - \frac{w}{2} + \frac{w^2}{2L}\right)$$

$$A_2 = \int_{-\frac{L}{2}}^{w} z_2 \, ds = \int_{-\frac{L}{2}}^{w}\left(\frac{L}{2} - w\right)\left(\frac{1}{2} + \frac{x}{L}\right)\frac{1}{I_c\frac{ds}{dx}}\, ds = \int_{\frac{L}{2}}^{w}\left(\frac{L}{2} - w\right)\left(\frac{1}{2} + \frac{x}{L}\right)\frac{dx}{I_c}$$

$$A_2 = \frac{1}{I_c}\left(\frac{L}{2} - w\right)\left(\frac{x}{2} + \frac{x^2}{2L}\right) = \frac{1}{I_c}\left(\frac{L}{2} - w\right)\left(\frac{w}{2} + \frac{L}{4} + \frac{w^2}{2L} - \frac{L^2}{8L}\right)$$

$$A_2 = \frac{1}{I_c}\left(\frac{L}{2} - w\right)\left(\frac{L}{8} + \frac{w}{2} + \frac{w^2}{2L}\right)$$

$$A_p = A_1 + A_2 = \frac{1}{I_c}\left[\frac{L}{2}\left(\frac{L}{4} + \frac{w^2}{L}\right) + w\,(-w)\right] = \frac{1}{I_c}\left[\frac{L^2}{8} + \frac{w^2}{2} - w^2\right]$$

$$A_p = \frac{1}{2I_c}\left[\frac{L^2}{4} - w^2\right] \qquad (21)$$

Referring to Fig. 8, the area for the $\frac{M}{EI}$ diagram of the springing moment at the right hand end will next be found. The equation of the plane is:

$$z = \frac{M_R}{I_x}\left(\frac{1}{2} + \frac{x}{L}\right)$$

$$A_{M_R} = \int_{-\frac{L}{2}}^{\frac{L}{2}} z \, ds = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{M_R}{I_c\frac{ds}{dx}}\left(\frac{1}{2} + \frac{x}{L}\right) ds = \frac{M_R}{I_c}\int_{-\frac{L}{2}}^{\frac{L}{2}}\left(\frac{1}{2} + \frac{x}{L}\right) dx$$

$$A_{M_R} = \frac{M_R}{I_c}\left(\frac{x}{2} + \frac{x^2}{2L}\right) = \frac{M_R}{I_c}\left(\frac{L}{4} + \frac{L}{4} + \frac{L^2}{8L} - \frac{L^2}{8L}\right) = \frac{M_R L}{2I_c} \qquad (22)$$

Similarly, the area of the $\frac{M}{EI}$ diagram for the moment at the left hand springing is:

$$A_{M_L} = \frac{M_L\, L}{2I_c} \qquad (23)$$

Referring to Fig. 9, the area for the $\frac{M}{EI}$ diagram of the horizontal thrust will be found. The equation of the plane is:

$$z = \frac{H}{I_x}\left(h - \frac{x^2}{K}\right)$$

$$A_H = 2\int_0^{\frac{L}{2}} z\,ds = 2\int_0^{\frac{L}{2}} \frac{H}{I_c\frac{ds}{dx}}\left(h - \frac{x^2}{K}\right)ds = \frac{2H}{I_c}\int_0^{\frac{L}{2}}\left(h - \frac{x^2}{K}\right)dx$$

$$A_H = \frac{2H}{I_c}\left(hx - \frac{x^3}{3K}\right) = \frac{2H}{I_c}\left(\frac{hL}{2} - \frac{L^3}{24K}\right) = \left(\frac{2H}{I_c}\right)\left(\frac{hL}{2} - \frac{L^3}{24\frac{L^2}{4h}}\right)$$

$$A_H = \frac{2H}{I_c}\left(\frac{hL}{2} - \frac{hL}{6}\right) = \frac{2}{3}\frac{HhL}{I_c} \tag{24}$$

Step two in the development of the method is the derivation of equations for the centroid distances of these four areas. The procedure is as follows:

<u>UNIT</u> <u>LOAD</u>. For the $\frac{M}{EI}$ diagram of the unit load, "d" is the centroid distance along the X-axis, and "r" is the centroid distance along the Y-axis.

$$d = \bar{x} = \frac{M_p'}{A_p} = \frac{1}{A_p}\left[\int_w^{\frac{L}{2}} xz_1ds + \int_{-\frac{L}{2}}^{w} xz_2ds\right]$$

Integrating, substituting limits, and simplifying, the result is:

$$d = \frac{\frac{w}{6I_c}\left(\frac{L^2}{4} - w^2\right)}{\frac{1}{2I_c}\left(\frac{L^2}{4} - w^2\right)} = \frac{w}{3} \tag{25}$$

To find the centroid distance along the Y-axis:

$$r = \bar{y} = \frac{M_p''}{A_p} = \frac{1}{A_p}\left[\int_{w}^{\frac{L}{2}} yz_1 ds + \int_{-\frac{L}{2}}^{w} yz_2 ds\right]$$

Integrating, substituting limits, and simplifying,

the result is:

$$r = \frac{\frac{h}{3L^2 I_c}\left(\frac{L^4}{16} - w^4\right)}{\frac{1}{2I_c}\left(\frac{L^2}{4} - w^2\right)} = \frac{2h}{3L^2}\left(\frac{L^2}{4} + w^2\right) \tag{26}$$

<u>MOMENT</u> <u>AT</u> <u>THE</u> <u>SPRINGING</u>. For the $\frac{M}{EI}$ diagram of the moment at the right springing(and also for the left springing) "c" is the centroid distance along the X-axis, and "m" is the centroid distance along the Y-axis.

$$c = \frac{M'_{M_R}}{A_{M_R}} = \frac{1}{A_{M_R}}\int_{-\frac{L}{2}}^{\frac{L}{2}} xz\ ds = \frac{1}{A_{M_R}}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{M_R}{I_c}\ x\left(\frac{1}{2} + \frac{x}{L}\right) dx$$

Integrating, substituting limits, and simplifying,

the result is:

$$c = \frac{\frac{M_R L^2}{12 I_c}}{\frac{M_R L}{2I_c}} = \frac{L}{6} \tag{27}$$

To find the centroid distance along the Y-axis:

$$m = \frac{M''_{M_R}}{A_{M_R}} = \frac{1}{A_{M_R}}\int_{-\frac{L}{2}}^{\frac{L}{2}} yz\ ds = \frac{1}{A_{M_R}}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{M_R}{I_c}\ \frac{x^2}{K}\left(\frac{1}{2} + \frac{x}{L}\right) dx$$

Integrating, substituting limits, and simplifying,

the result is:

$$m = \frac{\frac{M_R Lh}{6I_c}}{\frac{M_R L}{2I_c}} = \frac{h}{3} \qquad (28)$$

HORIZONTAL THRUST. For the $\frac{M}{EI}$ diagram of the horizontal thrust, the centroid distance along the X-axis is 0, and the centroid distance along the Y-axis is "n".

$$n = \frac{M''_H}{A_H} = \frac{1}{A_H}\, 2\int_0^{\frac{L}{2}} yz\ ds = \frac{2}{A_H}\int_0^{\frac{L}{2}} \frac{x^2}{K}\,\frac{H}{I_c}\left(h - \frac{x^2}{K}\right)dx$$

Integrating, substituting limits, and simplifying, the result is:

$$n = \frac{\frac{2Hh^2L}{15I_c}}{\frac{2HhL}{3I_c}} = \frac{h}{5} \qquad (29)$$

The third step in the development of the method is the substitution of the values for the various areas and centroid distances in the equilibrium equations that resulted from the conjugate structure. Since the solution of the conjugate structure equations is the same whether the moment of inertia is constant or varying, equations (15), (16), and (17) will again be used here. The simplicity of the expressions for the areas and the centroid distances allows direct substitution of them in the equilibrium equations as follows:

$$(15) \qquad A_H = A_p\left(\frac{m-r}{m-n}\right)$$

Substituting the proper expressions:

$$\frac{2}{3}\,\frac{HhL}{I_c} = \frac{1}{2I_c}\left[\frac{L^2}{4} - w^2\right]\left[\frac{\frac{h}{3} - \frac{2h}{3L^2}\left(\frac{L^2}{4} + w^2\right)}{\frac{h}{3} - \frac{h}{5}}\right]$$

$$H = \frac{1}{4hL}\left[\frac{L^2}{4} - w^2\right]\left[\frac{1 - \frac{2}{L^2}\left(\frac{L^2}{4} + w^2\right)}{\frac{1}{3} - \frac{1}{5}}\right]$$

Letting $v = \frac{w}{L}$:

$$H = \frac{1}{4hL}\left[\frac{L^2}{4} - v^2L^2\right]\left[\frac{1 - \frac{2}{L^2}\left(\frac{L^2}{4} + v^2L^2\right)}{\frac{2}{15}}\right]$$

$$H = \frac{15L}{8h}\left[\frac{1}{8} - v^2 + 2v^4\right] \qquad (30)$$

To find the equation for the moment at the right springing:

$$(16)\quad A_{M_R} = \frac{A_p}{2}\left(\frac{m-r}{m-n} + \frac{d}{c} - 1\right)$$

$$\frac{M_R L}{2I_c} = \frac{1}{2}\,\frac{1}{2I_c}\left[\frac{L^2}{4} - w^2\right]\left[\frac{\frac{h}{3} - \frac{2h}{3L^2}\left(\frac{L^2}{4} + w^2\right)}{\frac{h}{3} - \frac{h}{5}} + \frac{\frac{w}{3}}{\frac{L}{6}} - 1\right]$$

Substituting $v = \frac{w}{L}$ and simplifying:

$$M_R = \frac{L}{4}\left[\frac{1}{8} + v - 3v^2 - 4v^3 + 10v^4\right] \qquad (31)$$

To find the equation for the moment at the left springing:

$$(17)\quad A_{M_L} = \frac{A_p}{2}\left(\frac{m-r}{m-n} - \frac{d}{c} - 1\right)$$

Substituting the proper values as above, and simplifying:

$$M_L = \frac{L}{4}\left[\frac{1}{8} - v - 3v^2 + 4v^3 + 10v^4\right] \qquad (32)$$

Formulas (30), (31), and (32) then are the actual equations of the influence lines for the horizontal thrust and springing moments. Again caution is called to the fact that equations (31) and (32) are VALID ONLY WHEN THE UNIT LOAD IS PLACED TO THE LEFT OF CENTER OR AT THE CENTER. When the load is to the right of center, INTERCHANGE M_R and M_L in equations (31) and (32).

These three equations can easily be converted to a form containing only the quantities "L" and "w" by substituting $\frac{w}{L}$ for "v". However, when dealing with ten load points, as is often the case, "v" becomes 0.1, 0.2, 0.3, 0.4, and 0.5 in succession, affording convenient numbers with which to work.

Only five points, or half the span need be computed, since the values of "H" are symmetrical about the center-line, and the values of M_R and M_L are merely interchanged for the values of the springing moments on the opposite side of the span.

If it be desired to work out the values of the influence line ordinates from one end rather than from the center, the following expressions can be used:

$$H = \frac{15}{4}\,\frac{bu^2}{h}(1-u)$$

$$M_A = \frac{bu}{2}(7u-5u^2-2)$$

$$M_B = \frac{bu^2}{2}(3-5u)$$

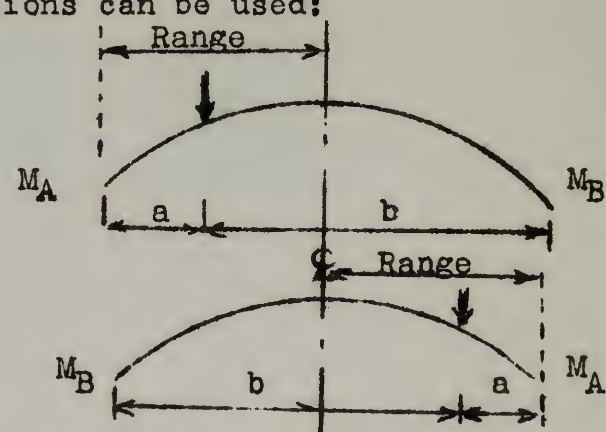
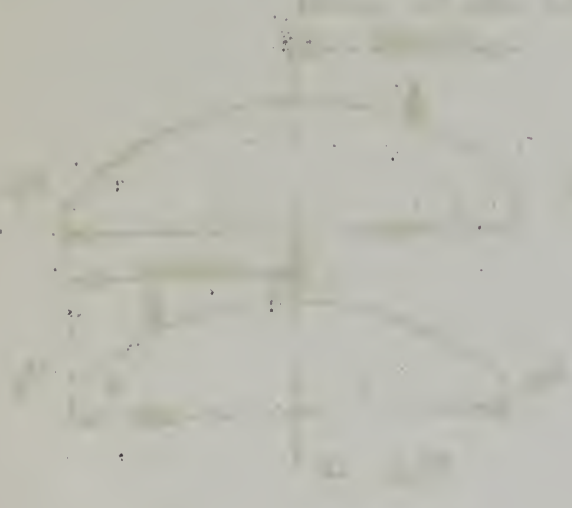


Fig. 12



where $u = \frac{a}{L}$ and M_A is on the same side of the centerline of the arch as "a". Again these equations are valid only for one half of the span, and the proper range must be used, as shown in Fig. 12.

RELATIONSHIPS BETWEEN DIFFERENT ARCHES

In PART I, it was shown that certain relationships exist between arches having the same ratio of rise to span. However, with the moment of inertia varying as assumed in PART II, it can be easily seen that for the same load point (1) THE THRUST VARIES DIRECTLY AS THE LENGTH OF SPAN AND INVERSELY AS THE RISE, and (2) THE VALUE OF THE SPRINGING MOMENTS VARIES DIRECTLY AS THE LENGTH OF SPAN, REGARDLESS OF THE RISE.

CONSTANTS FOR THE UNIT ARCH ($I_x = I_c \sec\alpha$)

Load Point	Fixed End Moment Right	Left	Thrust
1	.01125L	-.06073L	.0304L/h
2	.03200L	-.06400L	.0920L/h
3	.04725L	-.03675L	.1652L/h
4	.04800L	.00000	.2160L/h
5	.03125L	.03125L	.2344L/h
6	.00000	.04800L	.2160L/h
7	-.03675L	.04725L	.1652L/h
8	-.06400L	.03200L	.0920L/h
9	-.06073L	.01125L	.0304L/h

APPLICATIONS OF VARYING I FORMULAS

For given rise to span ratios, the unit values of H, M_R, and M_L as computed for a parabolic arch having a cross-sectional moment of inertia that is constant, are compared with the values for an arch having a moment of inertia that varies according to the same function as that used in this thesis.

Rise to Span Ratio	Load Point	H Const. I	H Vary. I	M_R Const. I	M_R Vary. I	M_L Const. I	M_L Vary. I
	1	0.77	0.76	.0116	.0113	-.0604	-.0605
	2	2.41	2.40	.0325	.0320	-.0633	-.0640
0.04	3	4.14	4.14	.0478	.0473	-.0359	-.0367
	4	5.40	5.40	.0487	.0480	.0008	.0000
	5	5.86	5.86	.0320	.0317	.0320	.0317
	1	0.39	0.38	.0118	.0113	-.0600	-.0605
	2	1.21	1.20	.0326	.0320	-.0627	-.0640
0.08	3	2.07	2.07	.0477	.0473	-.0354	-.0367
	4	2.70	2.70	.0484	.0480	.0010	.0000
	5	2.93	2.93	.0319	.0317	.0319	.0317
	1	0.16	0.15	.0117	.0113	-.0589	-.0605
	2	0.49	0.48	.0318	.0320	-.0603	-.0640
0.20	3	0.83	0.83	.0460	.0473	-.0334	-.0367
	4	1.07	1.08	.0463	.0480	.0013	.0000
	5	1.16	1.17	.0303	.0317	.0303	.0317
	1	0.11	0.10	.0116	.0113	-.0579	-.0605
	2	0.33	0.32	.0308	.0320	-.0585	-.0640
0.30	3	0.55	0.55	.0440	.0473	-.0321	-.0367
	4	0.71	0.72	.0434	.0480	.0012	.0000
	5	0.76	0.78	.0282	.0317	.0282	.0317
	1	0.085	0.076	.0116	.0113	-.0569	-.0605
	2	0.253	0.240	.0309	.0320	-.0562	-.0640
0.40	3	0.416	0.414	.0432	.0473	-.0305	-.0367
	4	0.528	0.540	.0423	.0480	.0016	.0000
	5	0.566	0.586	.0276	.0317	.0276	.0317

Comparing the values resulting from the solution with the constant I with those from the variable I, it is seen that for rise to span ratios up to about 0.20 the values from the two sources check within 5%. From this information, it can be concluded that using the simple equations for the variable I, reasonably close design values for the thrust and the springing moments can be obtained with practically no effort at all. Above 0.20 and up to about 0.40 the values of the thrust check within 3%, so these values can be used with reliance. Above 0.20 the values for the springing moments show quite a variation from the true values, so that the designer who wishes to work close to his allowable stress, should use the exact solution outlined in PART I of this thesis.

A second valuable application of the short formulas (30), (31), and (32) is that they can be used to obtain a close check on work with parabolic arches whose moment of inertia is constant.

A third application of these formulas lies in the exact solution parabolic arches whose moment of inertia varies according to the same function as that used in this paper. Since many arches are of this character, this application may prove most valuable.

CONCLUSION

The exact solution developed in this thesis represents a new method of attack in the solution the parabolic arch. Previous methods made use of the various approximate solutions, lengthy and tedious in their application. This approach, together with the equations given, permits a comparatively rapid solution for the required influence lines, and one that is exact as well.

When dealing with an arch whose moment of inertia varies such that $I_x = I_c \sec \alpha$, the same function as that used in PART II of this thesis, the solution becomes extremely simple and rapid.

In addition to the formulas for the arch with the varying moment of inertia, a table of values for a span of unit length and unit rise has been included, so that the exact values of the thrust and springing moments can be obtained merely by applying these constants to the arch at hand.

It has also been pointed out that the short formulas can be employed to check the values as obtained by another solution, to obtain directly values within 5% of the exact ones when the ratio of rise to span is less than 0.20, and to obtain directly from these formulas the value of the horizontal thrust for an arch of constant I within 3% of the true value when the ratio of rise to span is 0.40 or less.

With the basic approach as outlined in this thesis, there is further work that can be done in the complete development of the exact method of solution for arch structures.

The authors suggest the following subjects for future theses:

(1) The exact solution of the unsymmetrical parabolic arch, with (a) constant moment of inertia, and (b) varying moment of inertia.

(2) The exact solution of the symmetrical (and unsymmetrical) arch with moments of inertia that vary according to functions different from the one used here, and the comparison of results with those given here.

(3) The exact solution of elliptical arches.

In the opinion of the authors the exact method as developed in this thesis opens a new field of approach in the solution of indeterminate structures whose present approximate solutions are lengthy and laborious. It is sincerely hoped that further development along these lines will shorten and simplify the solution of many indeterminate structures.